Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **55**, 1-18; **http://www.math.u-szeged.hu/ejqtde/**

Periodic solutions to a p-Laplacian neutral Duffing equation with variable parameter

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Abstract. We study a type of p-Laplacian neutral Duffing functional differential equation with variable parameter to establish new results on the existence of T-periodic solutions. The proof is based on a famous continuation theorem for coincidence degree theory. Our research enriches the contents of neutral equations and generalizes known results. An example is given to illustrate the effectiveness of our results.

Keywords: variable parameter, neutral, coincidence degree theory

MSC 2000: 34B15, 34B24, 34B20

Supported by The Science Foundation of Educaion Department of Guangxi Province(No. 201012MS025), Youth PhD Development Fund of Central University of Finance and Economics 121 Talent Cultivation Project (NO.QBJZH201004) and Discipline Construction Fund of Central University of Finance and Economics.

1 Introduction

Neutral functional differential equations (in short NFDEs) are more wider and complicated than retarded equations. Such equations depend on past as well as present values but which involve derivatives with delays as well as the function itself. J. Hale [1] studied the following NFDE(D, f):

$$\frac{d}{dt}D(t, x_t) = f(t, x_t),$$

where D is a difference operator for NFDE(D, f). In order to guarantee continuation of the solution operator $T(t, \sigma, \varphi)$, Hale gave an important concept: Suppose $D : C \to \mathbb{R}^n$ is linear and atomic at 0 and let $C_D = \{\phi \in C : D\phi = 0\}$. The operator D is said to to be stable if the zero solution of the homogeneous "difference" equation

$$Dy_t = 0, \quad t \geq 0, \; y_0 = \psi \in C_D$$

is uniformly asymptotically stable. Thus one can study NFDEs by using the similar methods belonging to retarded equations under the condition of D is stable, see [2]-[6]. But when the operator D is not stable, how can we study existence and stability of solutions to NFDEs, which is very important for theory and applications. To best our knowledge, when the operator D is not stable, there are few results on the existence of solutions to NFDEs. In 1995, under the non-resonance condition, we can only find that Zhang [7] studied a kind of neutral differential system and relieved the stability restriction. Zhang gave some properties for the difference operator A and obtained the following results: Define the operator A on C_T

$$A : C_T \to C_T, [Ax](t) = x(t) - cx(t - \tau), \forall t \in \mathbb{R},$$

where $C_T = \{x : x \in C(\mathbb{R}, \mathbb{R}), x(t + T) \equiv x(t)\}$, c is a constant. when $|c| \neq 1$, then A has a unique continuous bounded inverse A^{-1} satisfying

$$[A^{-1}f](t) = \begin{cases} \sum\limits_{j \geq 0} c^j f(t - j\tau), & \text{if } \; |c| < 1, \quad \forall f \in C_T, \\ -\sum\limits_{j \geq 1} c^{-j} f(t + j\tau), & \text{if } \; |c| > 1, \quad \forall f \in C_T. \end{cases}$$

After that, Based on [7], Lu [8] gave some inequalities for A:

(1) $||A^{-1}|| \leq \frac{1}{|1-|k||}$;

(2) $\int_0^T |[A^{-1}f](t)|dt \leq \frac{1}{|1-|k||} \int_0^T |f(t)|dt, \forall f \in C_T$;

(3) $\int_0^T |[A^{-1}f](t)|^2 dt \leq \frac{1}{|1-|k||} \int_0^T |f(t)|^2 dt, \forall f \in C_T$.

On the basis of work of Zhang and Lu, many authors obtained existence results of periodic solutions to different kinds of NFDEs. For example, in [9], the authors investigated a second-order neutral equation with multiple deviating arguments:

$$\frac{d^2}{dt^2}(x(t) - kx(t-\tau)) = f(x(t))x'(t) + \alpha(t)g(x(t)) + \sum_{j=1}^{n} \beta_j(t)g(x(t-\gamma_j(t))) + p(t)$$

Liu and Huang [10] studied the following NFDE:

$$(u(t) + Bu(t-\tau))' = g_1(t, u(t)) - g_2(t, u(t-\tau_1)) + p(t).$$

But, when c is a variable $c(t)$, there are no corresponding results for A. In 2009, when c is a variable $c(t)$, we obtained the properties of the neutral operator $A : C_T \to C_T, \quad [Ax](t) = x(t) - c(t)x(t-\tau)$ in [11]. Using the results of [11], we have obtained some existence results for first-order and second-order neutral equations with variable parameter. At present, we note that $p-$Laplacian neutral equations have attracted much attention from researchers. In [12]-[13], Zhu and Lu studied the following p-Laplacian NFDEs:

$$\left(\varphi_p[(x(t) - cx(t-\sigma))']\right)' + g(t, x(t-\tau(t))) = e(t)$$

and

$$\left(\varphi_p[(x(t) - cx(t-\sigma))']\right)' = f(x(t))x'(t) + \Sigma_{j=1}^{n}\beta_j(t)g(x(t-\gamma_j(t))) + p(t).$$

However, there have been few results for the existence of periodic solutions to p-Laplacian neutral equations for the cases of a variable $c(t)$. The reasons for it lie in the following three aspects. The first is that the differential operator $\varphi_p(u) = |u|^{p-2}u, \ p \neq 2$ is no longer linear, so the theory of coincidence degree can not been used directly and verifying $L-$compact is difficult; the second is that an a priori bound of solutions is not easy to estimate; finally, the second condition of Mawhin's continuation theorem is not easy to verify. So in this paper we will overcome these difficulties and obtain the existence of periodic solutions to equation (1.1) by constructing proper projections $P,\ Q$ and some skills of inequalities.

In this paper, we consider the p-Laplacian neutral Duffing functional differential equation with variable parameter of the form:

$$(\varphi_p((x(t)-c(t)x(t-\tau))'))' + g(x(t-\gamma(t))) = e(t), \tag{1.1}$$

where $\varphi_p : \mathbb{R} \to \mathbb{R},\ \varphi_p(u) = |u|^{p-2}u,\ p > 1;\ g \in C(\mathbb{R}, \mathbb{R});\ c,\ \gamma,\ e$ are continuous $T-$periodic functions defined on $\mathbb{R}$; τ is a given constant.

2 Main Lemmas

In this section, we give some notations and lemmas which will be used in this paper. Let

$$c_0 = \max_{t\in[0,T]} |c(t)|,\quad \sigma = \min_{t\in[0,T]} |c(t)|,\quad c_1 = \max_{t\in[0,T]} |c'(t)|,$$

$$C_T = \{x | x \in C(\mathbb{R}, \mathbb{R}),\ x(t+T) \equiv x(t),\ \forall t \in \mathbb{R}\}$$

with the norm

$$|\varphi|_0 = \max_{t\in[0,T]} |\varphi(t)|,\quad \forall \varphi \in C_T$$

and

$$C_T^1 = \{x | x \in C^1(\mathbb{R}, \mathbb{R}),\ x(t+T) \equiv x(t),\ \forall t \in \mathbb{R}\}$$

with the norm

$$||\varphi|| = \max_{t\in[0,T]} \{|\varphi|_0,\ |\varphi'|_0\},\quad \forall \varphi \in C_T^1.$$

Clearly, C_T and C_T^1 are Banach spaces. Define linear operators:

$$A : C_T \to C_T,\quad [Ax](t) = x(t) - c(t)x(t-\tau),\quad \forall t \in \mathbb{R}.$$

Lemma 2.1. *[11] If $|c(t)| \neq 1$, then operator A has continuous inverse A^{-1} on C_T, satisfying*

(1)

$$[A^{-1}f](t) = \begin{cases} f(t) + \sum\limits_{j=1}^{\infty} \prod\limits_{i=1}^{j} c(t-(i-1)\tau)f(t-j\tau), \ c_0 < 1, \ \forall f \in C_T, \\ -\frac{f(t+\tau)}{c(t+\tau)} - \sum\limits_{j=1}^{\infty} \prod\limits_{i=1}^{j+1} \frac{1}{c(t+i\tau)} f(t+j\tau+\tau), \ \sigma > 1, \ \forall f \in C_T. \end{cases}$$

(2)

$$\int_0^T |[A^{-1}f](t)|dt \leq \begin{cases} \frac{1}{1-c_0} \int_0^T |f(t)|dt, \ c_0 < 1, \ \forall f \in C_T, \\ \frac{1}{\sigma-1} \int_0^T |f(t)|dt, \ \sigma > 1, \ \forall f \in C_T. \end{cases}$$

Let X and Y be real Banach spaces and let $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L. This means that ImL is closed in Y and $dimKerL = codimImL < +\infty$. If L is a Fredholm operator with index zero, then there exist continuous projectors $P : X \to X$, $Q : Y \to Y$ such that $ImP = KerL$, $ImL = KerQ = Im(I-Q)$. It follows that $L_{D(L)\cap KerP} : (I-P)X \to ImL$ is invertible. Denote by K_p the inverse of L_P.

Let Ω be an open bounded subset of X, a map $N : \bar{\Omega} \to Y$ is said to be L-compact in $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and the operator $K_p(I-Q)N(\bar{\Omega})$ is relatively compact. Because ImQ is isomorphic to $KerL$, there exists an isomorphism $J : ImQ \to KerL$. We first recall the famous Mawhin's continuation theorem.

Lemma 2.2. *[14] Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$, is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \bar{\Omega} \to Y$ is L-compact on $\bar{\Omega}$. if all the following conditions hold:*

(1) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \forall \lambda \in (0,1)$,

(2) $Nx \notin ImL, \forall x \in \partial\Omega \cap KerL$,

(3) $deg\{JQN, \Omega \cap KerL, 0\} \neq 0$,

where $J : ImQ \to KerL$ is an isomorphism. Then equation $Lx = Nx$ has a solution on $\bar{\Omega} \cap D(L)$.

In order to use Mawhin's continuation theorem to obtain the existence of T-periodic solutions of the equation (1.1), we rewrite the equation (1.1) in the form of the two-dimensional differential system

$$\begin{cases} (Ax_1)'(t) = \varphi_q(x_2(t)), \\ x_2'(t) = -g(x_1(t-\gamma(t))) + e(t), \end{cases} \tag{2.1}$$

where $q > 1$ is a constant with $\frac{1}{p} + \frac{1}{q} = 1$. Obviously if $x(t) = (x_1(t), x_2(t))^T$ is a T-periodic solution to system (2.1), then $x_1(t)$ must be a T-periodic solution to equation (1.1). Thus, in order to prove that equation (1.1) has a T-periodic solution, it suffices to show that system (2.1) has a T-periodic solution. Now we set

$$X = \{x = (x_1(\cdot), x_2(\cdot))^T \in C(\mathbb{R}, \mathbb{R}^2) |\ x(t+T) \equiv x(t)\}$$

with the norm $||x|| = \max\{|x_1|_0, |x_2|_0\}$. Equipped with the above norm $||\cdot||$, X is Banach space. Meanwhile, let

$$L : D(L) \subset X \to X,\ Lx = \begin{pmatrix} (Ax_1)' \\ x_2' \end{pmatrix}, \tag{2.2}$$

$$N : X \longrightarrow X,\ (Nx)(t) = \begin{pmatrix} \varphi_q(x_2(t)) \\ -g(x_1(t-\gamma(t))) + e(t) \end{pmatrix}, \tag{2.3}$$

where $D(L) = \{x : x \in C^1(\mathbb{R}, \mathbb{R}^2) |\ x(t+T) = x(t)\}$. We get

$$ImL = \left\{ y | y \in X, \int_0^T y(s)ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

Since for all $x \in KerL$, $(x_1(t) - c(t)x_1(t-\tau))' = 0$, then

$$x_1(t) - c(t)x_1(t-\tau) = 1. \tag{2.4}$$

Let $\phi(t)$ be the unique $T-$periodic solution of (2.4), then $\phi(t) \neq 0$ and

$$KerL = \left\{ \begin{pmatrix} a\phi(t) \\ \\ a \end{pmatrix}, a \in \mathbb{R} \right\}.$$

Obviously, ImL is a closed in X and $\mathrm{dimKer}L = \mathrm{codimIm}L = 1$. Hence L is a Fredholm operator with index zero. Define continuous projectors $P,\ Q$

$$P : X \to \mathrm{Ker}L,\ (Px)(t) = \begin{pmatrix} \frac{\int_0^T x_1(t)\phi(t)dt}{\int_0^T \phi^2(t)dt}\phi(t) \\ \frac{1}{T}\int_0^T x_2(t)dt \end{pmatrix}$$

and

$$Q : X \to X/\mathrm{Im}L,\ Qy = \begin{pmatrix} \frac{1}{T}\int_0^T y_1(t)dt \\ \\ \frac{1}{T}\int_0^T y_2(t)dt \end{pmatrix}.$$

Hence

$$ImP = KerL, \quad KerQ = ImL.$$

Let

$$L_P = L|_{D(L)\cap \mathrm{Ker}P} : D(L) \cap \mathrm{Ker}P \to \mathrm{Im}L,$$

then

$$L_P^{-1} = K_p : \mathrm{Im}L \to D(L) \cap \mathrm{Ker}P.$$

Since $\mathrm{Im}L \subset C_T$ and $D(L) \cap \mathrm{Ker}P \subset C_T^1$, so K_p is an embedding operator. Hence K_p is a completely operator in $\mathrm{Im}L$. By the definitions of Q and N, it knows that $QN(\bar{\Omega})$ is bounded on $\bar{\Omega}$, here Ω is a bounded open set on X. Hence nonlinear operator N is L-compact on $\overline{\Omega}$.

3 Main results

For the sake of convenience, we list the following conditions.

(H_1) There is a constant $D > 0$ such that

$$\begin{cases} g(x) < -|e|_0 & for\ x > D, \\ g(x) > |e|_0 & for\ x < -D. \end{cases}$$

(H$_2$) There is a constant r such that

$$\limsup_{x\to-\infty}\frac{|g(x)|}{|x|^{p-1}}\le r\in[0,\infty).$$

Theorem 3.1. *Suppose that $\int_0^T\phi^2(t)dt\neq 0$, $\int_0^T e(t)dt=0$, $|c(t)|\neq 1$ and assumptions (H$_1$), (H$_2$) are all satisfied, then equation (1.1) has at least one T-periodic solution, if*

$$\max\{\tfrac{c_1T}{1-c_0},\ \tfrac{2(1+c_0)rT^p}{(1-c_0-c_1T)^p}\}<1 \qquad for\ c_0<\tfrac{1}{2},$$

$$\max\{\tfrac{c_1T}{\sigma-1},\ \tfrac{2(1+c_0)rT^p}{(\sigma-1-c_1T)^p}\}<1 \qquad for\ \sigma>1.$$

Proof. Consider the following operator equation:

$$Lx=\lambda Nx,\quad \lambda\in(0,1),$$

where L and N are are defined by (2.2) and (2.3), respectively. Let

$$\Omega_1=\{x|x\in D(L),Lx=\lambda Nx,\lambda\in(0,1)\}.$$

If $x=\begin{pmatrix}x_1\\ x_2\end{pmatrix}\in\Omega_1$, then x must satisfy

$$\begin{cases}(Ax_1)'(t)=\lambda\varphi_q(x_2(t)),\\ x_2'(t)=-\lambda g(x_1(t-\gamma(t)))+\lambda e(t).\end{cases}\tag{3.1}$$

From the first equation of (3.1), we get $x_2(t)=\varphi_p(\frac{1}{\lambda}(Ax_1)'(t))$, combining with the second equation of (3.1) yields

$$(\varphi_p((Ax_1)'(t)))'+\lambda^pg(x_1(t-\gamma(t)))=\lambda^pe(t).\tag{3.2}$$

Let t_0 be the point, where Ax_1 achieves its maximum on $[0,T]$, i.e.,

$$(Ax_1)(t_0)=\max_{t\in[0,T]}(Ax_1)(t).$$

Then $(Ax_1)'(t_0)=0$ and $x_2(t_0)=\varphi_p(\frac{1}{\lambda}(Ax_1)'(t_0))=0,\forall\lambda\in(0,1)$. We claim

$$x_2'(t_0)\le 0.\tag{3.3}$$

In fact, if $x_2'(t_0) > 0$, then there exists a constant $\delta > 0$ such that $x_2'(t) > 0$ for $t \in [t_0, t_0 + \delta]$, then $x_2(t) > x_2(t_0) = 0$, for $t \in [t_0, t_0 + \delta]$. So $(Ax_1)'(t) = \lambda \varphi_q(x_2(t)) > 0$ for $t \in [t_0, t_0 + \delta]$ and thus $(Ax_1)(t) > (Ax_1)(t_0)$, which contradicts the assumption of t_0. This proves (3.3). From the second equation of (3.1), we have

$$-\lambda g(x_1(t_0 - \gamma(t_0))) + \lambda e(t_0) \leq 0,$$

then

$$g(x_1(t_0 - \gamma(t_0))) \geq -|e|_0.$$

By assumption (H_1),

$$x_1(t_0 - \gamma(t_0)) \leq D. \tag{3.4}$$

Integrating both sides of (3.2) over [0,T], we get

$$\int_0^T g(x_1(t - \gamma(t)))dt = 0. \tag{3.5}$$

From integral mean value theorem and (3.5), we know that there exists a constant $t_1 \in [0, T]$ such that

$$g(x_1(t_1 - \gamma(t_1))) = 0.$$

Assumption (H_1) implies

$$x_1(t_1 - \gamma(t_1)) \geq -D. \tag{3.6}$$

From (3.4) and (3.6), it is easy to prove that there exists a constant $\xi \in [0, T]$ such that

$$|x_1(\xi)| \leq D. \tag{3.7}$$

In fact, by (3.4) we know $x_1(t_0 - \gamma(t_0)) \in [-D, D]$, or $x_1(t_0 - \gamma(t_0)) < -D$.

(1) If $x_1(t_0 - \gamma(t_0)) \in [-D, D]$. Let $t_0 - \gamma(t_0) = kT + \xi,\ k \in \mathbb{Z},\ \xi \in [0, T]$. This proves (3.7).

(2) If $x_1(t_0 - \gamma(t_0)) < -D$, from (3.6) and the fact that the $x_1(t)$ is continuous on $\mathbb{R}$, there is a

point t_2 between $t_0-\gamma(t_0)$ and $t_1-\gamma(t_1)$ such that $|x_1(t_2)| \le D$. Let $t_2 = kT+\xi,\ k \in \mathbb{Z},\ \xi \in [0,T]$. This also proves (3.7). Hence we get

$$|x_1|_0 = \max_{t\in[0,T]} |x_1(\xi) + \int_\xi^t x_1'(s)ds| \le |x_1(\xi)| + \int_0^T |x_1'(s)|ds \le D + \int_0^T |x_1'(s)|ds. \tag{3.8}$$

Let

$$E_1 = \{t \in [0,T] : x_1(t-\gamma(t)) < -\rho\},\quad E_2 = \{t \in [0,T] : |x_1(t-\gamma(t))| \le \rho\},$$

$$E_3 = \{t \in [0,T] : x_1(t-\gamma(t)) > \rho\},$$

where $\rho > D > 0$ is a given constant. Integrating the two sides of (3.2) on $[0,T]$, we get

$$\int_0^T g(x_1(t-\gamma(t)))dt = 0.$$

Therefore, using (H_1) and (H_2), we obtain

$$\begin{aligned}\int_{E_3} |g(x_1(t-\gamma(t)))|dt &= -\int_{E_3} g(x_1(t-\gamma(t)))dt\\ &= \int_{E_1\cup E_2} g(x_1(t-\gamma(t)))dt\\ &\le \int_{E_1\cup E_2} |g(x_1(t-\gamma(t)))|dt.\end{aligned} \tag{3.9}$$

Since $\frac{2(1+c_0)rT^p}{(1-c_0-c_1T)^p} < 1$, there exists a constant $\varepsilon > 0$ such that

$$\frac{2(1+c_0)(r+\varepsilon)T^p}{(1-c_0-c_1T)^p} < 1. \tag{3.10}$$

For such ε, by assumption (H_2), there exists a constant $\rho > 0$ such that

$$|g(u)| \le (r+\varepsilon)|u|^{p-1} \quad \text{for } u < -\rho. \tag{3.11}$$

From (3.9) and (3.11), we get

$$\begin{aligned}\int_0^T |g(x_1(t-\gamma(t)))|dt &= \int_{E_1\cup E_2\cup E_3} |g(x_1(t-\gamma(t)))|dt\\ &\le 2\int_{E_1\cup E_2} |g(x_1(t-\gamma(t)))|dt\\ &\le 2(r+\varepsilon)T|x_1|_0^{p-1} + 2Tg_\rho,\end{aligned} \tag{3.12}$$

where $g_\rho = \max_{t\in E_2} |g(x_1(t-\gamma(t)))|$. On the other hand, multiplying the two sides of equation (3.2) by $(Ax_1)(t)$ and integrating them over $[0,T]$, combining with (3.12), then

$$\begin{aligned}\int_0^T |(Ax_1)'(t)|^p dt &\le (1+c_0)|x_1|_0\Big(\int_0^T |(g(x_1(t-\gamma(t)))|dt + T|e|_0\Big)\\ &\le (1+c_0)|x_1|_0 \int_0^T |g(x_1(t-\gamma(t)))|dt + (1+c_0)|x_1|_0 T|e|_0 \\ &\le 2(1+c_0)(r+\varepsilon)T|x_1|_0^p + (1+c_0)(2g_\rho T + T|e|_0)|x_1|_0.\end{aligned} \tag{3.13}$$

For simplicity, let $k_1 = 2(1+c_0)(r+\varepsilon)T,\ k_2 = (1+c_0)(2g_\rho T + T|e|_0)$. From (3.8) and (3.13), we have

$$\begin{aligned}\int_0^T |(Ax_1)'(t)|^p dt &\le k_1|x_1|_0^p + k_2|x_1|_0\\ &\le k_1\left(D + \int_0^T |x_1'(t)|dt\right)^p + k_2\int_0^T |x_1'(t)|dt + Dk_2.\end{aligned} \tag{3.14}$$

From $[Ax_1](t) = x_1(t) - c(t)x_1(t-\tau)$, $\forall x_1 \in C_T^1$, we have

$$(Ax_1')(t) = (Ax_1)'(t) + c'(t)x_1(t-\tau),$$

then from Lemma 2.1 and (3.8), if $c_0 < \frac{1}{2}$ we have

$$\begin{aligned}\int_0^T |x_1'(t)|dt &= \int_0^T |(A^{-1}Ax_1')(t)|dt\\ &\le \int_0^T \tfrac{|(Ax_1')(t)|}{1-c_0}dt\\ &= \int_0^T \tfrac{|(Ax_1)'(t)+c'(t)x_1(t-\tau)|}{1-c_0}dt\\ &\le \int_0^T \tfrac{|(Ax_1)'(t)|}{1-c_0}dt + \tfrac{c_1T}{1-c_0}\left(D + \int_0^T |x_1'(t)|dt\right).\end{aligned}$$

In view of $\frac{c_1T}{1-c_0} < 1$, we have

$$\begin{aligned}\int_0^T |x_1'(t)|dt &\le \int_0^T \tfrac{|(Ax_1)'(t)|}{1-c_0-c_1T}dt + \tfrac{c_1TD}{1-c_0-c_1T}\\ &\le \tfrac{T^{\frac{1}{q}}}{1-c_0-c_1T}\left(\int_0^T |(Ax_1)'(t)|^p dt\right)^{\frac{1}{p}} + \tfrac{c_1TD}{1-c_0-c_1T}\end{aligned} \tag{3.15}$$

Case 1. If $\int_0^T |(Ax_1)'(t)|dt = 0$, then $\int_0^T |x_1'(t)|dt \le \frac{c_1TD}{1-c_0-c_1T}$, by (3.8),

$$|x_1|_0 \le D + \frac{c_1TD}{1-c_0-c_1T}. \tag{3.16}$$

Case 2. If $\int_0^T |(Ax_1)'(t)|dt > 0$. By (3.14) and (3.15), we have

$$
\begin{aligned}
\int_0^T |(Ax_1)'(t)|^p dt &\le k_1\left(D+\int_0^T |x_1'(t)|dt\right)^p + k_2\int_0^T |x_1'(t)|dt + Dk_2 \\
&\le k_1\left(D+\int_0^T \tfrac{|(Ax_1)'(t)|}{1-c_0-c_1T}dt + \tfrac{c_1TD}{1-c_0-c_1T}\right)^p \\
&+k_2\int_0^T \tfrac{|(Ax_1)'(t)|}{1-c_0-c_1T}dt + \tfrac{k_2c_1TD}{1-c_0-c_1T} + Dk_2 \\
&= k_1\left(\tfrac{D-Dc_0}{1-c_0-c_1T} + \int_0^T \tfrac{|(Ax_1)'(t)|}{1-c_0-c_1T}dt\right)^p \\
&+k_2\int_0^T \tfrac{|(Ax_1)'(t)|}{1-c_0-c_1T}dt + \tfrac{k_2c_1TD}{1-c_0-c_1T} + Dk_2.
\end{aligned}
\tag{3.17}
$$

Clearly,

$$
\begin{aligned}
&\left(\tfrac{D-Dc_0}{1-c_0-c_1T} + \tfrac{\int_0^T |(Ax_1)'(t)|dt}{1-c_0-c_1T}\right)^p \\
&= \tfrac{1}{(1-c_0-c_1T)^p}\left(\int_0^T |(Ax_1)'(t)|dt\right)^p \left(1+\tfrac{D-Dc_0}{\int_0^T |(Ax_1)'(t)|dt}\right)^p.
\end{aligned}
\tag{3.18}
$$

By classical elementary inequalities, there is a constant $h(p) > 0$ which is dependent on p only, such that

$$
(1+u)^p < 1 + (1+p)u, \forall u \in (0, h(p)]. \tag{3.19}
$$

If $\frac{D-Dc_0}{\int_0^T |(Ax_1)'(t)|dt} > h$, then $\int_0^T |(Ax_1)'(t)|dt < \frac{D-Dc_0}{h}$. By (3.8) and (3.15),

$$
\begin{aligned}
|x_1|_0 &< D + \int_0^T |x_1'(t)|dt \\
&\le \int_0^T \tfrac{|(Ax_1)'(t)|}{1-c_0-c_1T}dt + \tfrac{c_1TD}{1-c_0-c_1T} + D \\
&< \tfrac{D-Dc_0}{h(1-c_0-c_1T)} + \tfrac{D-Dc_0}{1-c_0-c_1T} \\
&= \tfrac{(h+1)(D-Dc_0)}{h(1-c_0-c_1T)}.
\end{aligned}
\tag{3.20}
$$

If $\frac{D-Dc_0}{\int_0^T |(Ax_1)'(t)|dt} \le h$. By (3.18) and (3.19), then

$$
\begin{aligned}
&\left(\frac{D-Dc_0}{1-c_0-c_1T} + \frac{\int_0^T |(Ax_1)'(t)|dt}{1-c_0-c_1T}\right)^p \\
&\le \frac{1}{(1-c_0-c_1T)^p}\left(\int_0^T |(Ax_1)'(t)|dt\right)^p \left(1+\frac{(p+1)(D-Dc_0)}{\int_0^T |(Ax_1)'(t)|dt}\right) \\
&\le \frac{\left(\int_0^T |(Ax_1)'(t)|dt\right)^p}{(1-c_0-c_1T)^p} + \frac{(p+1)(D-Dc_0)}{(1-c_0-c_1T)^p}\left(\int_0^T |(Ax_1)'(t)|dt\right)^{p-1}.
\end{aligned}
\tag{3.21}
$$

By (3.17) and (3.21),

$$\begin{aligned}\int_0^T |(Ax_1)'(t)|^p dt \le{}& \frac{k_1}{(1-c_0-c_1T)^p}\left(\int_0^T |(Ax_1)'(t)|dt\right)^p + \frac{k_1(p+1)(D-Dc_0)}{(1-c_0-c_1T)^p}\left(\int_0^T |(Ax_1)'(t)|dt\right)^{p-1}\\ &+ k_2\int_0^T \frac{|(Ax_1)'(t)|}{1-c_0-c_1T}dt + \frac{k_2c_1TD}{1-c_0-c_1T} + Dk_2\\ \le{}& \frac{k_1}{(1-c_0-c_1T)^p}T^{\frac{p}{q}}\int_0^T |(Ax_1)'(t)|^p dt\\ &+ \frac{k_1(p+1)(D-Dc_0)}{(1-c_0-c_1T)^p}T^{\frac{p-1}{q}}\left(\int_0^T |(Ax_1)'(t)|^p dt\right)^{\frac{p-1}{p}}\\ &+ \frac{k_2T^{\frac{1}{q}}}{1-c_0-c_1T}\left(\int_0^T |(Ax_1)'(t)|^p dt\right)^{\frac{1}{p}} + \frac{k_2c_1TD}{1-c_0-c_1T} + Dk_2.\end{aligned} \tag{3.22}$$

In view of the definition the number k_1, from (3.10), (3.22), $\frac{p-1}{p} < 1$ and $\frac{1}{p} < 1$, there is a constant $M_1 > 0$ such that $\int_0^T |(Ax_1)'(t)|^p dt \le M_1$. It follows from (3.15) that

$$\int_0^T |x_1'(t)|dt \le \frac{T^{\frac{1}{q}}M_1^{\frac{1}{p}}}{1-c_0-c_1T} + \frac{c_1TD}{1-c_0-c_1T} := M_2.$$

By (3.8) we get

$$|x_1|_0 \le D + M_2. \tag{3.23}$$

Consequently, from (3.16), (3.20) and (3.23), we have

$$|x_1|_0 \le \max\{D + \frac{c_1TD}{1-c_0-c_1T}, \frac{(h+1)(D-Dc_0)}{h(1-c_0-c_1T)}, D+M_2\} := M_3.$$

If $\sigma > 1$, from the conditions of Theorem 3.1, similar to the above proof, we also obtain that there exists a constant $M_4 > 0$ such that

$$|x_1|_0 \le M_4.$$

Then we have

$$|x_1|_0 < \max\{M_3,\ M_4\} + 1 := \bar{M}.$$

In view of the first equation of (3.1) we have $\int_0^T |x_2(t)|^{q-2}x_2(t)dt = 0$. From integral mean value theorem, there exists a constant $\eta \in [0, T]$ such that $x_2(\eta) = 0$. Hence $|x_2|_0 \le \int_0^T |x_2'(t)|dt$.

By the second equation of (3.1) we get

$$\int_0^T |x_2'(t)|dt \le \int_0^T |g(x_1(t-\gamma(t)))|dt + \int_0^T |e(t)|dt$$

$$\le Tg_{\bar{M}} + T|e|_0,$$

where $g_{\bar{M}} = \max_{|u|<\bar{M}} |g(u)|$. So we obtain

$$|x_2|_0 \le g_{\bar{M}} + T|e|_0 := \widetilde{M}.$$

We have proved that if $x = (x_1, x_2)^T \in D(L),\ Lx = \lambda Nx,\ \lambda \in (0,1)$, then $|x_1|_0 \le \bar{M}$ and $|x_2|_0 \le \widetilde{M}$. Let $M = \max\{\bar{M},\ \widetilde{M}\}$ and $\Omega = \{x = (x_1, x_2)^T \in X : |x_1|_0 \le M, |x_2|_0 \le M\}$. Then $M > D$ and it is clear that the condition (1) of Lemma 2.2 is satisfied. Moreover, for any $x = (x_1, x_2)^T \in X$, we have

$$QNx = \begin{pmatrix} \frac{1}{T}\int_0^T \varphi_q(x_2(t))dt \\ -\frac{1}{T}\int_0^T g(x_1(t-\gamma(t)))dt \end{pmatrix}.$$

Since $KerL = (a\phi(t), a)^T$, where $a \in \mathbb{R}$ and $ImL = KerQ$, if $QNx = 0$ for some $x = (x_1, x_2)^T \in \partial\Omega \cap KerL$, then $x_2 \equiv 0,\ x_1 = a\phi(t)$, and

$$\int_0^T g(a\phi(t))dt = 0. \tag{3.24}$$

When $c_0 < \frac{1}{2}$, we have

$$\begin{aligned}\phi(t) = A^{-1}(1) \quad &= 1 + \sum_{j=1}^{\infty}\prod_{i=1}^{j} c(t-(i-1)\tau) \\ &\ge 1 - \sum_{j=1}^{\infty}\prod_{i=1}^{j} c_0 \\ &= 1 - \tfrac{c_0}{1-c_0} \\ &= \tfrac{1-2c_0}{1-c_0} := \delta_1 > 0.\end{aligned}$$

Then we have

$$a \le \frac{D}{\delta_1}.$$

Otherwise, $\forall t \in [0,T],\ a\phi(t) > D$, from assumption ($H_1$), we have

$$\int_0^T g(a\phi(t))dt < 0$$

which is contradiction to (3.24). When $\sigma > 1$, we have

$$\begin{aligned}\phi(t) = A^{-1}(1) &= -\tfrac{1}{c(t+\tau)} - \sum_{j=1}^{\infty}\prod_{i=1}^{j+1}\tfrac{1}{c(t+i\tau)} \\ &\leq -\tfrac{1}{\sigma} - \sum_{j=1}^{\infty}\prod_{i=1}^{j+1}\tfrac{1}{\sigma} \\ &= -\tfrac{1}{\sigma-1} := \delta_2 < 0.\end{aligned}$$

Then we have

$$a \leq -\frac{D}{\delta_2}.$$

Otherwise, $\forall t \in [0,T]$, $a\phi(t) < -D$, from assumption (H1), we have

$$\int_0^T g(a\phi(t))dt > 0$$

which is contradiction to (3.24). One has $|x_1|_0 = \max\{\frac{D}{\delta_1}, -\frac{D}{\delta_2}\}|\phi|_0 = M \leq D$, which is a contradiction. So $QNx \neq 0$ for all $x \in \partial\Omega \cap KerL$ and thus the condition (2) of Lemma 2.2 is satisfied. It remains to verify the condition (3) of Lemma 2.2. In order to prove it, let

$$J : ImQ \to KerL, \quad J(x_1, x_2)^T = (x_2, x_1)^T,$$

and $H(x,\mu) = \mu x + (1-\mu)JQNx$ for $(x,\mu) \in X \times [0,1]$. Then we have

$$H(x,\mu) = \begin{pmatrix} \mu x_1 - \frac{(1-\mu)}{T}\int_0^T g(x_1(t-\gamma(t)))dt \\ \mu x_2 + \frac{(1-\mu)}{T}\int_0^T \varphi_q(x_2(t))dt \end{pmatrix}.$$

It is not difficult to verify that, using (H_1), for any $x \in \partial\Omega \cap KerL$ and $\mu \in [0,1]$, we have $H(x,\mu) \neq 0$. Therefore,

$$\begin{aligned}\deg\{JQN, \Omega \cap KerL, 0\} &= \deg\{H(\cdot,0), \Omega \cap KerL, 0\} \\ &= \deg\{H(\cdot,1), \Omega \cap KerL, 0\} \\ &= \deg\{I, \Omega \cap KerL, 0\} \\ &\neq 0.\end{aligned}$$

Therefore, by using Lemma 2.2, we see that the equation $Lx = Nx$ has a solution $x = (x_1, x_2)^T$ in $\bar{\Omega}$, i. e., the equation (1.1) has a $T-$periodic solution x_1.

□

Remark 3.1. When $\frac{1}{2} \leq c_0 < 1$, we can not obtain the existence results of periodic solutions for equation (1.1). This is an interesting problem for further research.

As an application, we consider the following NFDE:

$$(\varphi_3((x(t) - 0.1(2 - \cos t)x(t - \tau))'))' + g(x(t - 1/2\sin t)) = \sin t, \tag{3.25}$$

where

$$g(u) = \begin{cases} -\frac{1}{10^8}u^2, & u > 10000, \\ -\frac{1}{10^4}u, & u \in [-10000, 10000], \\ \frac{1}{10^8}u^2, & u < -10000. \end{cases}$$

Clearly, the Eq. (3.25) is a particular case of (1.1) in which

$$p = 3,\ c(t) = 0.1(2 - \cos t),\ \gamma(t) = \frac{1}{2}\sin t,\ e(t) = \sin t.$$

Then we have $c_0 = 0.3 < \frac{1}{2}$, $c_1 = 0.1$, $T = 2\pi$ and $r = \frac{1}{10^8}$, and thus

$$\frac{c_1 T}{1 - c_0} = \frac{0.2\pi}{0.7} \approx 0.897 < 1$$

and

$$\frac{2(1 + c_0)rT^p}{(1 - c_0 - c_1 T)^p} = \frac{2.6 \times (2\pi)^3}{(0.7 - 0.2\pi)^3 \times 10^8} \approx 0.0187 < 1.$$

Here assumptions (H_1) and (H_2) are satisfied. By using Theorem 3.1, we know that equation (3.25) has at least one $2\pi-$periodic solution.

Acknowledgement

The author is very grateful to the referees for their helpful suggestions.

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(Received May 27, 2011)